

# Seminar 2

## Exercise 1

- 1) Is the equality  $\mathbb{P}(B|A) + \mathbb{P}(C|A) = \mathbb{P}(B \cup C|A)$  true?
- 2) Provide examples showing that the following equalities are generally not true:
  - a)  $\mathbb{P}(A|B \cup C) = \mathbb{P}(A|B) + \mathbb{P}(A|C)$
  - b)  $\mathbb{P}(B|A) + \mathbb{P}(B|\bar{A}) = 1$

## Exercise 2

Let there be a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and events  $H_1, H_2 \in \mathcal{F}$  have positive probabilities. Let us denote  $\mathbb{P}_{H_i} := \mathbb{P}(\cdot|H_i)$ ,  $i = 1, 2$ . Prove that

$$\mathbb{P}_{H_1}(\cdot|H_2) = \mathbb{P}(\cdot|H_1 \cap H_2) = \mathbb{P}_{H_2}(\cdot|H_1).$$

That is, for any  $A \in \mathcal{F}$ ,

$$\mathbb{P}_{H_1}(A|H_2) = \mathbb{P}(A|H_1 \cap H_2) = \mathbb{P}_{H_2}(A|H_1).$$

## Exercise 3

A deck of 52 cards is dealt to 4 players. One of the players announces that he has an ace.

- a. What is the probability that he has at least one more ace?
- b. What is the probability that he has at least one more ace if he announced that he has the ace of spades?

First, solve this problem directly, without using the concept of conditional probability (by redefining the set of elementary outcomes), and then by the definition of conditional probability.

#### Exercise 4

From the set  $1, 2, \dots, n$ , three distinct numbers are chosen in sequence without replacement. Find the conditional probability that the third number lies between the first and the second, given that the first number is less than the second.

#### Exercise 5

- a. There was one white ball and one black ball in a bag. One ball was drawn from it and placed in an empty box. Another white ball was also placed in the box. Finally, one ball was drawn from the box, and it turned out to be white. What is the probability that the remaining ball in the box is also white?
- b. Solve the previous problem assuming that initially there were 10 black and 7 white balls in the bag.

#### Exercise 6

(*Pólya's Urn Scheme*) An urn contains  $a$  white and  $b$  black balls. We perform  $n$  random draws, and immediately after each draw, the ball is returned to the urn along with  $m$  other balls of the same color. ( $m \geq -1$  and  $n \leq a + b$  if  $m = -1$ ).

- a. What is the probability that out of  $n = n_1 + n_2$  chosen balls,  $n_1$  will be white and  $n_2$  will be black?
- b. Prove that the probability of drawing a white ball on the  $i$ -th step is  $a/(a+b)$ .

#### Exercise 7

(*Monty Hall Paradox*). Imagine you are a contestant in a game where you have to choose one of three doors. Behind one of the doors is a car; behind the other two are goats. You choose one of the doors, for example, number 1. After this, the host, who knows where the car is and where the goats are, opens one of the remaining doors, for example, number 3, which has a goat behind it. He then asks you if you would like to change your choice to door number 2. Will your chances of winning the car increase if you accept the host's offer and change your choice? Clarifications: the car is placed behind any of the three doors with equal probability; the host knows where the car is; regardless of which door you choose, the host must in any case open a door with a goat (but not the one you chose) and offer to change the choice; if the host has a choice of which of the two doors to open (that is, you pointed to the correct door, and behind both remaining doors are goats), he chooses any of them with equal probability.

### Exercise 8

Covid is back in fashion! But British scientists are not asleep either: a new test has been developed with a sensitivity of 99% (i.e., it correctly diagnoses a sick person in 99% of cases) and a specificity of 99% (only 1% of healthy people are declared sick). It is known that in a certain happy village, 1 out of every 1000 inhabitants suffers from Covid. What is the probability that a resident of this village, who tested positive, is actually sick?

### Exercise 9

Agent D. is monitoring the movements of a company director. It is known that the director is in the office with a probability of 60% and at his dacha with a probability of 40%. Agent D. has two informants; the first is wrong with a probability of 20%, and the second with a probability of 10%. The first informant claims the director is in the office, while the second informant claims he is at the dacha. Where is the director?

### Additional Exercises

#### Exercise 10

Let  $n \geq 2$  and select a random number  $\xi$  from  $\{1, 2, \dots, n\}$ . Let  $A$  be the event on which  $\xi$  is even, and  $B$  the event on which  $\xi$  is divisible by 7. Find all  $n$  such that events  $A$  and  $B$  are independent.

#### Exercise 11

Events  $A, B, C$  are *pairwise* independent and equally likely,  $A \cap B \cap C = \emptyset$ . Find the maximum possible value of  $P(A)$ .

#### Exercise 12

Give an example of three events  $A, B, C$  that are pairwise independent but not independent. In general, give an example of  $n$  events  $A_1, \dots, A_n$  that are dependent but each subset of  $n - 1$  sets is independent.