Seminar 4

If it is written that a random variable ξ is "given" or "known", it is implied that its probability measure $\mathbb{P}_\xi$, CDF F_ξ, and PDF p_ξ (if the density exists) are known. If you are asked to find the distribution of ξ, it is sufficient to find any of the objects mentioned above.

Exercise 1

The distribution of a random vector is given:

$\beta \setminus \alpha$	-2	-1	0	1	2
-2	$1/32$	$1/32$	$1/24$	$1/32$	$1/32$
-1	$1/32$	$1/32$	$1/24$	$1/32$	$1/32$
0	$1/24$	$1/24$	$1/6$	$1/24$	$1/24$
1	$1/32$	$1/32$	$1/24$	$1/32$	$1/32$
2	$1/32$	$1/32$	$1/24$	$1/32$	$1/32$

a. Find the probability $P(\alpha = \beta)$.
b. Find the probability $P(\alpha > \beta)$.
c. Find the probability $P(\alpha \leq \beta)$.
d. Find the distributions of α and β.
e. Is it true that α and β are independent?
f. Find the distribution of $\alpha + \beta$.
g. Find the distribution of $\alpha\beta$.
h. Find the distribution of the random vector with components $\alpha + \beta$ and $\alpha - \beta$.
i. Find the distribution of the random vector with components $\alpha + \beta$ and $\alpha\beta$.

Solution

a. $\mathbb{P}(\alpha = \beta)$ is the sum of probabilities on the main diagonal:

$$\mathbb{P}(\alpha = \beta) = p_{-2,-2} + p_{-1,-1} + p_{0,0} + p_{1,1} + p_{2,2} = \frac{1}{32} + \frac{1}{32} + \frac{1}{6} + \frac{1}{32} + \frac{1}{32} = \frac{4}{32} + \frac{1}{6} = \frac{1}{8} + \frac{1}{6} = \frac{3+4}{24} =$$

b. $\mathbb{P}(\alpha > \beta)$ is the sum of all elements above the main diagonal. The table is symmetric, therefore $\mathbb{P}(\alpha > \beta) = \mathbb{P}(\alpha < \beta) = (1 - \mathbb{P}(\alpha = \beta))/2 = 17/48$.

c. $\mathbb{P}(\alpha \leq \beta) = 1 - \mathbb{P}(\alpha > \beta) = \frac{31}{48}$.

d. To get the distribution of α, sum over columns: e.g. $\mathbb{P}(\alpha = -2) = 4 \cdot \frac{1}{32} + \frac{1}{24} = \frac{1}{6}$. Due to the symmetry of the table, the distribution of β (sum over rows) is the same.

e. No, they are not independent, e.g. $\mathbb{P}(\alpha = -2, \beta = -2) = 1/32 \neq \mathbb{P}(\alpha = -2)\mathbb{P}(\beta = -2) = 1/36$.

To find the distributions of the next points, you need to group the cells of the table (these are tedious but straightforward calculations):

f. For $\alpha+\beta$: group the cells (i,j) with the same sum $k=i+j$.
g. For $\alpha\beta$: group the cells with the same product $k=i\cdot j$.
h. For $(\alpha+\beta,\alpha-\beta)$, this is in bijection with (α,β), so each entry corresponds to one entry in the table, i.e. $(\alpha,\beta)=((\alpha+\beta)+(\alpha-\beta),(\alpha+\beta)-(\alpha-\beta))/2$.
i. For $(\alpha+\beta,\alpha\beta)$, compute the joint law separating the cases where α or β vanishes.

Exercise 2 [H]

The distributions of independent random variables ξ and η are given:

ξ	-2	-1	0	1	2
	$1/4$	$1/8$	$1/4$	$1/8$	$1/4$

η	-2	-1	0	1	2
	$1/8$	$1/4$	$1/4$	$1/4$	$1/8$

a. Find the probability $P(\xi=\eta)$.
b. Find the probability $P(\xi>\eta)$.
c. Find the probability $P(\xi\le\eta)$.
d. Find the distributions of $\xi+\eta$ and $\xi-\eta$.
e. Find the distribution of $\xi\eta$.
f. Find the distribution of the random vector with components $\xi+\eta$ and $\xi-\eta$.
g. Are $\xi+\eta$ and $\xi-\eta$ dependent?
h. Find the distribution of the random vector with components $\xi+\eta$ and $\xi\eta$.

Solution

a. $\mathbb{P}(\xi=\eta)=\sum_k\mathbb{P}(\xi=k,\eta=k)=\sum_k\mathbb{P}(\xi=k)\mathbb{P}(\eta=k)=\frac{3}{16}$ due to independence.

Similarly b, c, d, e, f, h. The calculations are performed by considering $\mathbb{P}(\xi=i,\eta=j)=\mathbb{P}(\xi=i)\mathbb{P}(\eta=j)$ and summing the probabilities over the corresponding regions.

g. Let's check the independence of $U=\xi+\eta$ and $V=\xi-\eta$. $\mathbb{P}(U=4,V=0)=\mathbb{P}(\xi=2,\eta=2)\neq\mathbb{P}(V=0)\mathbb{P}(U=4)$. The variables U and V are dependent.

Independence of Random Variables

Exercise 3 [H]

Let $\xi\sim\text{Uniform}([0,1])$ and $\eta\sim\text{Bernoulli}(1/3)$. Define these random variables on the same probability space so that they are

a. independent
b. dependent.

This means that for each part, you need to devise a probability space and a random vector $\zeta = (\alpha, \beta)$ on it, such that $\mathbb{P}_\alpha = \mathbb{P}_\xi$ and $\mathbb{P}_\eta = \mathbb{P}_\beta$.

> **Solution**
>
> a. **Independent**: Take $\Omega = [0,1] \times [0,1]$ with the Lebesgue measure, as the probability space. Set $\alpha(t_1, t_2) = t_1$ and $\beta(t_1, t_2) = \mathbf{1}_{[0,1/3]}(t_2)$. We can also do the same taking $\Omega = [0,1] \times \{0,1\}$.
>
> b. **Dependent**: Take the space $\Omega = [0,1]$ with the Lebesgue measure. Set $\alpha(t) = t$, $\beta(t) = \mathbf{1}_{[0,1/3]}(t)$.

Exercise 4

Consider the probability space $([0,1], \mathcal{B}([0,1]), \text{Leb})$. Are the following random variables dependent?

a. $\xi(t) = 2t$, $\eta(t) = 1 - t^2$
b. $\xi(t) = \text{sign}\left[\sin(2\pi t)\right]$, $\eta(t) = \text{sign}\left[\sin(4\pi t)\right]$
c. $\xi(t) = \text{sign}\left[\sin(2\pi t)\right]$, $\eta(t) = \text{sign}(t - 1/3)$?

> **Solution**
>
> a. Dependent. They are functionally related: $\eta(t) = 1-(\xi(t)/2)^2$. If we know the value of $\xi(t)$, we uniquely know the value of $\eta(t)$. For instance for $\varepsilon > 0$ small enough, $\mathbb{P}(\xi \leq \varepsilon, \eta \leq \varepsilon) = 0 \neq \mathbb{P}(\xi \leq \varepsilon)\mathbb{P}(\eta \leq \varepsilon) > 0$.
> b. Independent. $\mathbb{P}(\xi = 1, \eta = 1) = 1/4 = \mathbb{P}(\xi = 1)\mathbb{P}(\eta = 1)$, and this is enough since both η and ξ only take two values with positive probability.
> c. Dependent. E.g. $\mathbb{P}(\xi = +1 | \eta = -1) = 1 > \mathbb{P}(\xi = +1)$.

Exercise 5 [H]

Provide an example of two discrete random variables that are dependent but not functionally dependent, or explain why such an example does not exist.

> **Solution**
>
> Two random variables α, β are called functionally dependent if one is a measurable function of the other, for example $\beta = f(\alpha)$. This means that knowing the value of α, we uniquely know the value of β. This is typically much stronger than just dependency. E.g. in the previous example $\xi(t) = \text{sign}\left[\sin(2\pi t)\right]$, $\eta(t) = \text{sign}(t-1/3)$ are dependent, but not functionally dependent.

Exercise 6 [H]

Is it true that if ξ and η are independent, then for any function $f : \mathbb{R} \to \mathbb{R}$, the random variables $f(\xi)$ and $f(\eta)$ are also independent?

> **Solution**
>
> We can even take f, g measurable. Then
>
> $$\mathbb{P}(f(\xi) \in A, g(\eta) \in B) = \mathbb{P}(\xi \in f^{-1}(A), \eta \in g^{-1}(B)) = \mathbb{P}(\xi \in f^{-1}(A))\mathbb{P}(\eta \in g^{-1}(B)) = \mathbb{P}(f(\xi) \in A)\mathbb{P}(g(\eta) \in B)$$

Exercise 7 [H]

Is it true that if ξ and η are dependent, then ξ^2 and η^2 must also be dependent?

> **Solution**
>
> In general it is not true that the independence of $f(\xi)$ and $g(\eta)$ implies the independence of ξ and η. E.g. take $\xi = \eta = 2\zeta - 1$, where $\zeta \sim \text{Bernoulli}(1/2)$. Then $\xi^2 = \eta^2 = 1$ are independent.

Exercise 8 [H]

Let the random vector (ξ, η) have a uniform distribution in the disk $(x, y) : x^2 + y^2 \leq 1$. Are the random variables ξ and η independent?

> **Solution**
>
> No, for instance for $\varepsilon > 0$ small enough, $\mathbb{P}(\xi \geq 1 - \varepsilon, \eta \geq 1 - \varepsilon) = 0 \neq \mathbb{P}(\xi \geq 1 - \varepsilon)\mathbb{P}(\eta \geq 1 - \varepsilon) > 0$.
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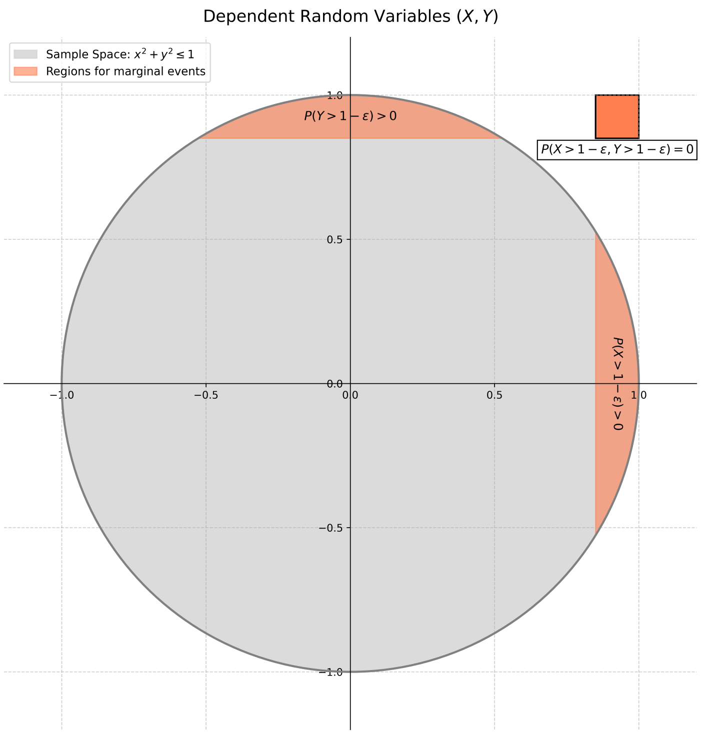
> 
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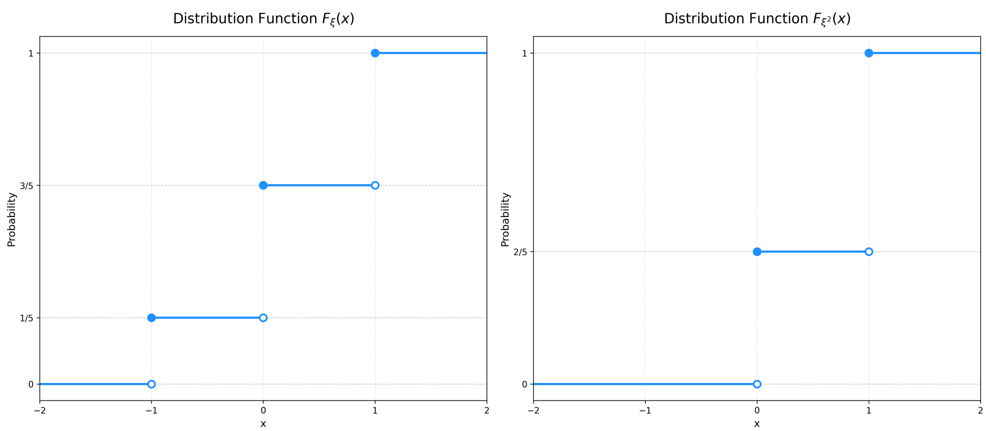
Distribution Function

Exercise 9

Consider a random variable ξ with the distribution below. Sketch F_ξ and F_{ξ^2}.

ξ	-1	0	1
$\mathbb{P}_\xi$	$1/5$	$2/5$	$2/5$

Solution



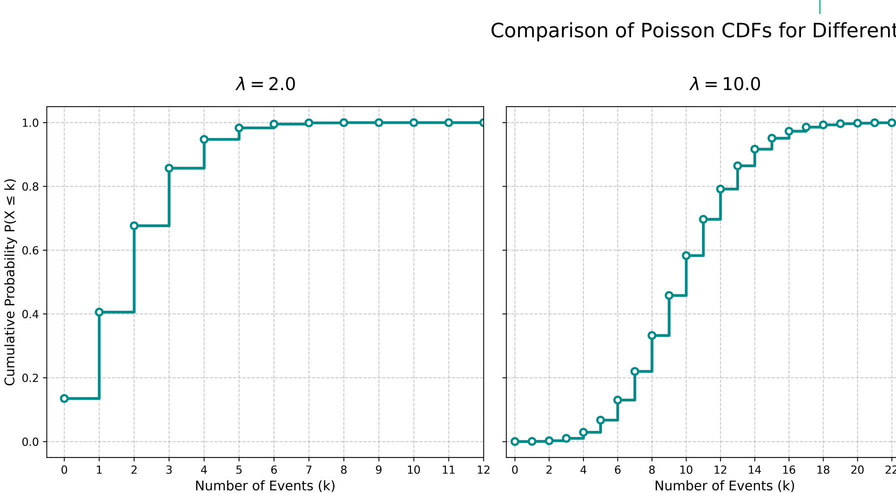
Exercise 10 [H]

Sketch the distribution function F_ξ for $\xi \sim \text{Poisson}(\lambda)$, $\lambda > 0$.

Solution

We have that:

- $F_\xi(x) = 0$ for $x < 0$.
- $F_\xi(x) = e^{-\lambda}$ for $0 \leq x < 1$.
- $F_\xi(x) = e^{-\lambda}(1 + \lambda)$ for $1 \leq x < 2$.



and so on. The function approaches 1 as $x \to \infty$.

Exercise 11

A rod of length 2 is broken at a random point. Explicitly define, specifying the probability space, the random variable ξ representing the length of the larger of the two resulting pieces. Find the CDFs F_ξ, F_{ξ^2} and the PDFs p_ξ, p_{ξ^2}.

Solution

Take $\Omega = [0, 2]$ with a uniform measure. The break point $U \sim \text{Uniform}([0, 2])$, the pieces have lengths U and $2 - U$. Thus we are interested in the random variable is $\xi = \max(U, 2 - U)$, which takes values in $[1, 2]$. Notice that $\xi \leq x \iff 2 - x \leq U \leq x$. Thus for $x \in [1, 2]$

$$F_\xi(x) = \int_{2-x}^{x} \frac{1}{2} du = x - 1$$

So ξ is uniform in $[1, 2]$, its density being constant in this interval.
The distribution function of ξ^2 is then $F_{\xi^2}(x) = \sqrt{x} - 1$ for $x \in [1, 4]$, its density being $(2x)^{-1/2}\mathbf{1}_{[1,4]}$.

Exercise 12

Given independent random variables $\xi_1, \dots, \xi_n$, find the distribution function of the random variable

a. $\max(\xi_1, \dots, \xi_n)$.
b. $\min(\xi_1, \dots, \xi_n)$.

Solution

Let $F_i(x) = \mathbb{P}(\xi_i \leq x)$ be the distribution function for ξ_i.

a. **Maximum**: Let $M_n = \max(\xi_1, \dots, \xi_n)$, then $\{M_n \leq x\}$ iff all the ξ_i are no greater than x. Thus by virtue of independence:

$$F_{M_n}(x) = \mathbb{P}(M_n \leq x) = \mathbb{P}(\xi_1 \leq x, \xi_2 \leq x, \dots, \xi_n \leq x) = \prod_{i=1}^{n} F_i(x)$$

If all ξ_i are identically distributed with CDF $F(x)$, then $F_{M_n}(x) = (F(x))^n$.

b. **Minimum**: $m_n = \min(\xi_1, \dots, \xi_n)$. In this case we reason as above on the complementary event $m_n > x$ to get $F_{m_n}(x) = 1 - \prod_{i=1}^{n}(1 - F_i(x))$. If all ξ_i are identically distributed with CDF $F(x)$, then $F_{m_n}(x) = 1 - (1 - F(x))^n$.

Exercise 13

Let the random variable ξ be continuous (i.e., its distribution function F_ξ is continuous). Find the distribution of the random variable $F_\xi(\xi)$.

Solution

Let us denote $F = F_\xi$, to recall that this is not a random function. Let $\eta = F(\xi)$. Since F is a distribution function, its values lie in the interval $[0, 1]$. Now $F^{-1}((-\infty, y])$ is an increasing (in y) closed subset of $\mathbb{R}$ of the form $F^{-1}((-\infty, y]) = (-\infty, s_y]$. Then, by the very definition of F

$$G(y) = \mathbb{P}(\eta \leq y) = \mathbb{P}(F(\xi) \leq y) = \mathbb{P}(\xi \in F^{-1}((-\infty, y])) = F(s_y)$$

If F is continuous, then $G(y) = F(s_y) = y$. Thus $F_\xi(\xi)$ is a uniform random variable. In general, if F is not continuous, $G(y) \leq y$.

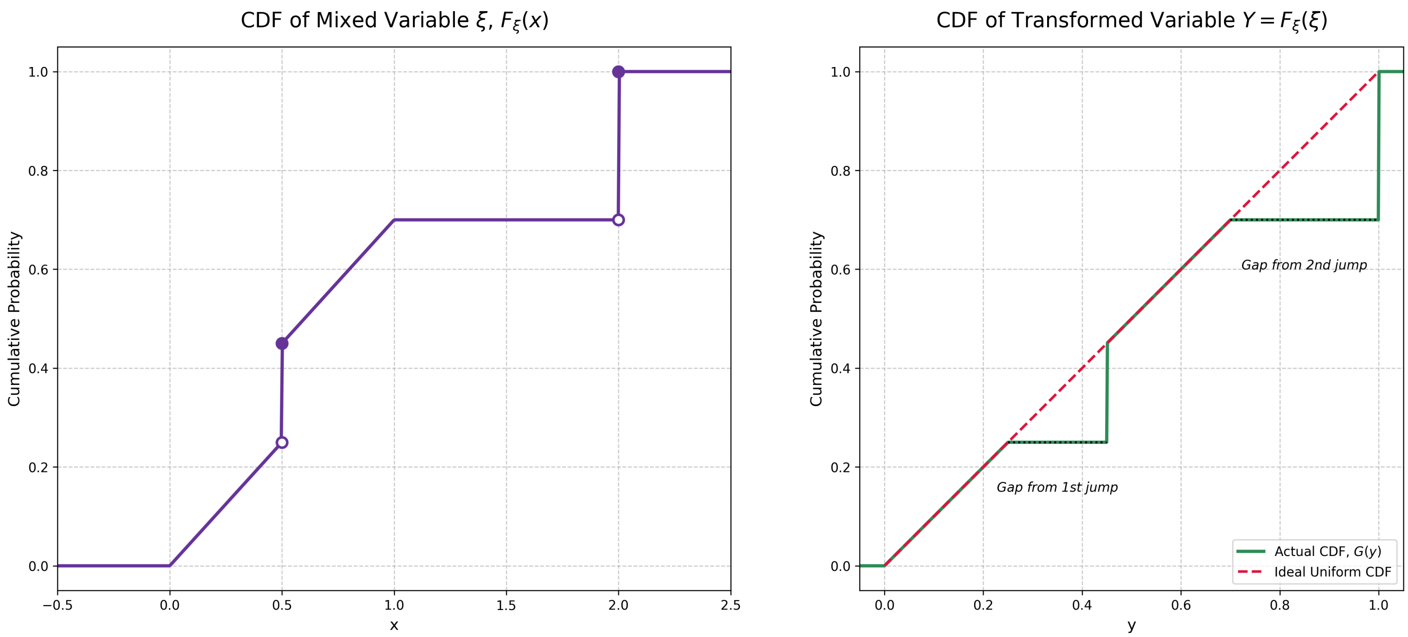


Figure 1

Exercise 14 [H]

Let $\xi \sim$ Uniform$([-1, 1])$. Find the distribution of the random variable $F_{|\xi|}(\xi)$.

Solution

$|\xi|$ is uniform in $[0, 1]$, thus $F_{|\xi|}(\xi) = \max(\xi, 0)$. Therefore

$$G(x) := \mathbb{P}(F_{|\xi|}(\xi) \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 + x/2 & \text{if } x \in [0, 1] \\ 1 & \text{if } x \geq 1 \end{cases}$$

Exercise 15*

(Mixtures). On a probability space Ω, the following construction depending on a family of random variables $\xi : \mathbb{R} \to \mathbb{R}$ is considered: first, using a random variable $\eta \sim$ Uniform$([0, 1])$, an interval $[0, \eta(\omega)]$ is chosen. Then, independently, a random variable $\zeta(\omega) := \xi_{\eta(\omega)}(\omega)$ is chosen with a given distribution on the interval $[0, \eta(\omega)]$. Find the probability density of the resulting random variable ξ, if

a. $\xi_a \sim$ Uniform$([0, a])$,
b. $\xi_a^2 \sim$ Uniform$([0, a])$.

> Solution

We can justify this formula either with measure theory, or using monotonicity in the law of ξ_a to have an explicit bound. In any case

$$\mathbb{P}(\zeta \le x) = \int_0^1 \mathbb{P}(\xi_a \le x)\, da$$

a. In this case the last formula gives for $x \in (0,1]$

$$\mathbb{P}(\zeta \le x) = \int_0^1 \tfrac{x}{a}\mathbf{1}_{[0,a)}(x) + \mathbf{1}_{[a,\infty)}(x)\, da = x - x\log(x)$$

which has density $-\log(x)\mathbf{1}_{(0,1]}(x)$.

b. In this case, for $x \in (0,1)$, $\mathbb{P}(\xi_a \le x) = \mathbb{P}(\xi_a^2 \le x^2) = x^2/a\mathbf{1}_{[0,\sqrt{a})}(x) + \mathbf{1}_{[\sqrt{a},\infty)}(x)$. Therefore for $x \in (0,1)$

$$\mathbb{P}(\zeta \le x) = \int_0^1 \tfrac{x^2}{a}\mathbf{1}_{[0,\sqrt{a})}(x) + \mathbf{1}_{[\sqrt{a},\infty)}(x) da = x^2 - 2x^2\log(x)$$

and the density is therefore $-4x\log(x)\mathbf{1}_{[0,1]}(x)$, a substantially different behavior around $x = 0$.

Exercise 16*

Let $\xi \sim \operatorname{Exp}(\lambda)$. Are its integer and fractional parts independent?

> Solution

They are independent. Let $X := \lfloor \xi \rfloor$ be the integer part, and $\eta := \xi - \lfloor \xi \rfloor$ be the fractional part. For $k \in \{0,1,2,\dots\}$ and $y \in [0,1)$, we compute the joint distribution

$$\mathbb{P}(X = k, \eta \le y) = \mathbb{P}(k \le \xi \le k+y) = e^{-\lambda k} - e^{-\lambda(k+y)} = e^{-\lambda k}(1 - e^{-\lambda y})$$

As this is the product of a function k times a function of y, the random variables are independent. We see in particular that X is geometric with (non-success) parameter $e^{-\lambda}$, while η has density $\lambda e^{-\lambda y}(1-e^{-\lambda})^{-1}\mathbf{1}_{[0,1]}(y)$.

Additional Exercises

Exercise 17

Let $\xi\colon \Omega \to E$ be a random variable, and $f\colon E \to F$ be measurable. Here E, F are measurable space. Prove that the $(\xi, f(\xi))$ are independent iff $f(\xi)$ is constant a.s.

> Solution

If $(\xi, f(\xi))$ are independent, then

$$\mathbb{P}(\xi \in A, f(\xi) \in B) = \mathbb{P}(\xi \in A, \xi \in f^{-1}(B)) = \mathbb{P}(\xi \in A)\mathbb{P}(\xi \in f^{-1}(B)) = \mathbb{P}(\xi \in A)\mathbb{P}(f(\xi) \in B)$$

If we now choose $A = f^{-1}(B)$ we get $\mathbb{P}(f(\xi) \in B) = \mathbb{P}(f(\xi) \in B)^2$. Namely $\mathbb{P}(f(\xi) \in B) \in \{0,1\}$ for all measurable B. This is the correct way of saying that $f(\xi)$ is constant. I.e., if the σ-algebra on $F contains singletons, this means that $f(\xi)$ is constant a.s.. Otherwise, *being constant* is not defined as a measurable event while saying that the law of $f(\xi)$ takes value in $\{0, 1\}$ still makes sense.