Seminar 5

Exercise 1

Let $\alpha \sim \text{Uniform}([0,1])$. Find the following functions:

a. The probability density function $p_\beta(x)$, if the random variable β is such that $\beta = 3\alpha - 1$.
b. The probability density function $p_\gamma(x)$, if the random variable γ is such that $\gamma = -\ln(\alpha)$.
c. The probability density function $p_\kappa(x)$, if the random variable κ is such that

$$\kappa = \begin{cases} 1 + \alpha + \alpha^2 + \ldots & \alpha \in (0,1) \\ 0 & \alpha \notin (0,1) \end{cases}$$

d. The probability density function $p_\epsilon(x)$, if

$$\epsilon = \begin{cases} \sum_{j=0}^{\infty} (-1)^j \alpha^j & \alpha \in (0,1) \\ 0 & \alpha \notin (0,1) \end{cases}$$

e. The cumulative distribution function $F_\rho(x)$, if the random variable ρ is such that

$$\rho = \begin{cases} 1 & \text{if } \alpha \text{ is irrational} \\ 0 & \text{if } \alpha \text{ is rational} \end{cases}$$

Solution

The density of α is $p_\alpha(t) = \mathbf{1}_{[0,1)}(t)$.

a. $\beta = 3\alpha - 1$. This is a linear transformation. $\alpha = (\beta + 1)/3$. The range for β is $[-1, 2]$. $p_\beta(x) = p_\alpha\left(\frac{x+1}{3}\right)\left|\frac{d\alpha}{d\beta}\right| = 1 \cdot \frac{1}{3} = \frac{1}{3}$ on $[-1, 2]$. This is $\text{Uniform}([-1, 2])$. In general, it is immediate to check that an affine transformation of a uniform random variable is uniform (on the interval being given by the same affine function).

b. $\gamma = -\ln(\alpha)$, $\alpha = e^{-\gamma}$. The range for γ is $[0, \infty)$. $p_\gamma(x) = p_\alpha(e^{-x})\left|\frac{d\alpha}{d\gamma}\right| = 1 \cdot |-e^{-x}| = e^{-x}$ on $[0, \infty)$. I.e. $\gamma \sim \exp(1)$.

c. $\kappa = \frac{1}{1-\alpha}$ for $\alpha \in (0,1)$. $\alpha = 1 - 1/\kappa$. The range for κ is $(1, \infty)$. $p_\kappa(x) = p_\alpha(1 - 1/x)\left|\frac{d\alpha}{d\kappa}\right| = 1 \cdot |1/x^2| = 1/x^2$ on $(1, \infty)$.

d. $\epsilon = \frac{1}{1+\alpha}$ for $\alpha \in (0,1)$, $\alpha = 1/\epsilon - 1$. The range for ϵ is $(1/2, 1)$. $p_\epsilon(x) = p_\alpha(1/x - 1)\left|\frac{d\alpha}{d\epsilon}\right| = 1 \cdot |-1/x^2| = 1/x^2$ on $(1/2, 1)$.

e. $\rho = 1$ a.s., thus $F_\rho(x) = \mathbf{1}_{[1,\infty)}(x)$.

Exercise 2

The random variable α is uniform on the interval $[-1,3]$, find the density of $-|\alpha|$.

Solution

If we visualize it graphically, $-|\alpha|$ pushes forward the density at points $x>0$ to the same at $-x$. The density of $-|\alpha|$ is this $\frac{1}{4}\mathbf{1}_{[-3,-1)}+\frac{1}{2}\mathbf{1}_{[-1,0)}$. We can also solve it analytically, indeed for $y<0$

$$f_{-|\alpha|}(y)=\frac{d}{dy}\mathbb{P}(-|\alpha|\leq y)=\frac{d}{dy}\mathbb{P}(\alpha\geq -y)+\frac{d}{dy}\mathbb{P}(\alpha\leq y)=\tfrac{1}{4}\mathbf{1}_{[-3,-1)}(y)+\tfrac{1}{2}\mathbf{1}_{[-1,0)}(y)$$

Exercise 3

The random variable α is uniform on the interval $[-1,3]$, find the cumulative distribution function for $\frac{|\alpha|}{\alpha}$.

Solution

Since $\mathbb{P}(\alpha=0)=0$, we have that $\frac{|\alpha|}{\alpha}$ is just the sign of α. So it -1 with probability $1/4$ and $+1$ with probability $3/4$. The distribution function is $\frac{1}{4}\mathbf{1}_{[-1,1)}+\mathbf{1}_{[1,\infty)}$.

Exercise 4

A random variable α is uniform on the interval $[-1,1]$, and a random variable β, independent of α, is a Bernoulli variable with parameter $p=\frac{1}{3}$.

a. Find the cumulative distribution function of the random variable $\alpha\beta$.
b. Find the cumulative distribution function of the random variable $|\alpha|\beta$.
c. Find the cumulative distribution function of the random variable $|2\alpha-1|\beta$.

Solution

a. $F_{\alpha\beta}(x)=\frac{x+1}{6}\mathbf{1}_{[-1,0)}(x)+\frac{x+5}{6}\mathbf{1}_{[0,1)}(x)+\mathbf{1}_{[1,\infty)}(x)$.
b. $F_{|\alpha|\beta}(x)=\frac{x+2}{3}\mathbf{1}_{[0,1)}(x)+\mathbf{1}_{[1,\infty)}$.
c. $F_{|2\alpha-1|\beta}(x)=\frac{4+x}{6}\mathbf{1}_{[0,1)}(x)+\frac{9+x}{12}\mathbf{1}_{[1,3)}(x)+\mathbf{1}_{[3,\infty)}(x)$.

Exercise 5

A random variable α is uniform on the interval $[0,1]$, and the random variable β is independent of α.

a. Find the probability density function of the random variable $2\alpha-\beta$, if β is distributed according to the exponential law with parameter 1.
b. Find the cumulative distribution function of the random variable $\alpha+\beta$, if β is discrete and distributed according to the Poisson law with parameter λ.
c. Find the cumulative distribution function of the random variable $\alpha+2\beta$, if β is a geometric random variable with parameter p.

Solution

a. Let $\xi = 2\alpha - \beta$. The density of 2α is $f_{2\alpha}(x) = \frac{1}{2}\mathbf{1}_{[0,2)}$. The density of $-\beta$ is $f_{-\beta}(x) = e^x \mathbf{1}_{(-\infty,0)}$. The density of the sum ξ is the convolution:

$$f_\xi(y) = \int_{-\infty}^{\infty} f_{2\alpha}(x) f_{-\beta}(y-x)dx = \frac{\mathbf{1}_{(-\infty,2)}(y)}{2} \int_{\max(0,y)}^{2} e^{y-x}dx = \begin{cases} \frac{e^y(1-e^{-2})}{2} & \text{if } y < 0. \\ \frac{1-e^{y-2}}{2} & \text{if } y \in [0,2). \\ 0 & \text{if } y \geq 2. \end{cases}$$

b. Let $\eta = \alpha + \beta$, then

$$F_\eta(x) = \sum_{k=0}^{\infty} \mathbb{P}(\eta \leq x | \beta = k)\mathbb{P}(\beta = k) = \sum_{k=0}^{\infty} \mathbb{P}(\alpha \leq x - k)\frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=0}^{\lfloor x \rfloor - 1} \frac{e^{-\lambda}\lambda^k}{k!} + (x - \lfloor x \rfloor)\frac{e^{-\lambda}\lambda^{\lfloor x \rfloor}}{\lfloor x \rfloor !}$$

c. Let $\zeta = \alpha + 2\beta$. $\mathbb{P}(\beta = k) = (1-p)^k p$ for $k = 0, 1, \ldots$. Thus $F_\zeta(x) = \sum_{k=0}^{\infty} \mathbb{P}(\alpha \leq x - 2k)(1-p)^k p$. So similarly to point b., F_ζ is piecewise affine, interpolating among the values of the $F_{2\beta}(x)$ at even integers points $x = 2k$.

Exercise 6

A random variable γ is distributed according to the exponential law with parameter a, a random variable θ is also distributed according to the exponential law with parameter b, and γ, θ are independent.

a. Find the probability density function of the r.v. $\sqrt{\gamma}$
b. Find the probability density function of the r.v. γ^2
c. Find the probability density function of the r.v. $1 - e^{-a\gamma}$
d. Find the probability density function of the r.v. $\max(\gamma, \theta)$
e. Find the probability density function of the r.v. $\min(\gamma, \theta)$
f. Find the probability density function of the r.v. $\gamma + \theta$

Solution

a. $F(y) = \mathbb{P}(\gamma \leq y^2) = 1 - e^{-ay^2}$ for $y \geq 0$. $p(y) = 2aye^{-ay^2}$.
b. $F(y) = \mathbb{P}(\gamma \leq \sqrt{y}) = 1 - e^{-a\sqrt{y}}$ for $y \geq 0$. $p(y) = \frac{a}{2\sqrt{y}}e^{-a\sqrt{y}}$.
c. $\zeta = 1 - e^{-a\gamma} = F_\gamma(\gamma)$. As we know, this transformation yields Uniform$([0,1])$ for any continuous random variables γ.
d. $F_{\max}(x) = F_\gamma(x)F_\theta(x) = 1 - e^{-ax} - e^{-bx} + e^{-(a+b)x}$. $p_{\max}(x) = ae^{-ax} + be^{-bx} - (a+b)e^{-(a+b)x}$ for $x \geq 0$.
e. $F_{\min}(x) = 1 - (1 - F_\gamma(x))(1 - F_\theta(x)) = 1 - e^{-(a+b)x}$. Namely the minimum is exponential of parameter $a + b$.
f. If $a \neq b$, $p_{\gamma+\theta}(y) = \frac{ab}{b-a}(e^{-ay} - e^{-by})$. If $a = b$, $p_{\gamma+\theta}(y) = a^2ye^{-ay}$.

Exercise 7*

Let $X_1, X_2 \ldots$ be independent random variables, with the same distribution $\exp(\lambda)$. Let $Y_n := \sum_{i=1}^{n} X_i$ and $N_t := \inf\{n \geq 0 : Y_{n+1} > t\}$, $t > 0$.

a. Prove that the distribution of Y_n has the density $\rho_n(y) := e^{-\lambda y}\frac{\lambda^n y^{n-1}}{(n-1)!}\mathbf{1}_{y\geq 0}$.
b. Prove that $\mathbb{P}(N_t = k) = e^{-\lambda t}(\lambda t)^k/k!$ (this means that $N_t \sim \text{Poisson}(\lambda t)$).

Solution

a. Proceed by induction.

- For $n = 1$, $Y_1 = X_1$, thus $\rho_1(y) = \lambda e^{-\lambda y}$.
- Assume the formula is true for n. $Y_{n+1} = Y_n + X_{n+1}$, a sum of the independent random variables, and the density of Y_{n+1} is the convolution of their densities:

$$\begin{aligned}\rho_{n+1}(y) &= \int_0^y \rho_n(x)\rho_1(y-x)dx = \int_0^y \left(e^{-\lambda x}\frac{\lambda^n x^{n-1}}{(n-1)!}\right)(\lambda e^{-\lambda(y-x)})dx \\ &= \frac{\lambda^{n+1}e^{-\lambda y}}{(n-1)!}\int_0^y x^{n-1}dx = e^{-\lambda y}\frac{\lambda^{n+1}y^n}{n!}\end{aligned}$$

b. Notice that $\rho'_{n+1} = -\lambda(\rho_{n+1} - \rho_n)$. Thus

$$\mathbb{P}(N_t = n) = \mathbb{P}(N_t < n+1) - \mathbb{P}(N_t < n) = \mathbb{P}(Y_{n+1} > t) - \mathbb{P}(Y_n > t) = \int_t^\infty \rho_{n+1}(y) - \rho_n(y)dy = -\int_t^\infty$$

which is the statement to be proved.

Exercise 8

A point (x, y) is chosen from the square $[0,1]\times[0,1]$ uniformly. Find the distribution of the random variables

a. x^2.
b. $x/(x+y)$.
c. $x^2 + y^2$.
d. $\min(x, y)$.
e. $\max(x, y)$.

Solution

a. Set $\xi := x^2$. Then $F_\xi(z) = \mathbb{P}(x^2 \leq z) = \sqrt{z}$ and $f_\xi(z) = 1/(2\sqrt{z})$.
b. Set $\xi = x/(x+y)$. ξ has the same law as $1-\xi = y/(x+y)$. So the density satisfies, $f_\xi(z) = f_\xi(1-z)$. Thus take $z \leq 1/2$, and notice

$$\mathbb{P}(\xi \leq z) = \mathbb{P}(x \leq zy/(1-z)) = z/(2(1-z))$$

since this is the area of a triangle with height 1 and base $z/(1-z)$. In particular the density is $f_\xi(z) = 2(1+|2z-1|)^{-2}$ for $z \in [0,1]$.
c. Set $\xi = x^2 + y^2$. For $z \in [0,1]$

$$F_\xi(z) = \mathbb{P}(x^2+y^2 \leq z) = \text{Area of quarter circle of radius } \sqrt{z} = \pi z/4$$

For $z \in (1,2]$, the set $\{x^2+y^2 \leq z\}$ is the union of two triangles and a circular sector. The triangles have height 1 and base $\sqrt{z}\sin(\arccos z^{-1/2})$. The circular sector spans an angle $\pi/2 - 2\arccos(z^{-1/2})$.

So if $z \in (1, 2]$

$$F_\xi(z) = \sqrt{z-1} + z(\pi/4 - \arccos z^{-1/2})$$

In particular $f_\xi(z) = \frac{\pi}{4} - \arccos(z^{-1/2})\mathbf{1}_{[1,2)}(z)$.

d. Set $\xi = \min(x, y)$. $F_\xi(z) = 1 - \mathbb{P}(\min(x, y) > z) = 1 - (1 - z)^2 = 2z - z^2$ and $f_\xi(z) = 2(1 - z)$.
e. Set $\xi = \max(x, y)$. $F_\xi(z) = z^2$ and $f_\xi(z) = 2z$.

Exercise 9

Let the random vector (α, β) be uniformly distributed in the region $\mathcal{G} = \{|x| + |y| < 1\}$. That is, the corresponding two-dimensional probability density is

$$f_{(\alpha,\beta)}(x, y) = \begin{cases} \text{const} & x, y \in \mathcal{G} \\ 0 & x, y \notin \mathcal{G} \end{cases} \tag{1}$$

a. What is the value of the constant in the formula?
b. Find the densities $f_\alpha(x)$, $f_\beta(y)$ of the distribution of the first coordinate α and the second coordinate β of the vector.
c. Are α and β dependent?
d. Find the probability densities for $\alpha + \beta$ and for $\alpha - \beta$.

Solution

a. The constant is $1/|\mathcal{G}| = 1/2$.
b. The density of α is obtained as pushforwarding the uniform measure on $\mathcal{G}$ on the segment $[-1, 1]$. So a graphical visualization immediately shows $f_\alpha(x) = f_\beta(x) = (1 - |x|)\mathbf{1}_{[-1,1]}$. We can also find this by computing $f_\alpha(x) = \int f_{\alpha,\beta}(x, y)dy$.
c. They are dependent, e.g. for $x \geq 1/2$, $\mathbb{P}(\alpha > x, \beta > x) = 0$, while $\mathbb{P}(\alpha > x) = \mathbb{P}(\beta > x) > 0$. More in general, we observe that if (α, β) are distributed as in Equation 1, they are independent iff $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ (up to a.e. equivalence) and α, β are uniform on $\mathcal{G}_1$, $\mathcal{G}_2$ respectively.
d. Let $U = \alpha + \beta, V = \alpha - \beta$. This is a rotation and scaling. The region $|x| + |y| < 1$ is transformed into the region $\mathcal{G}' = \{|u| < 1, |v| < 1\}$. The Jacobian of the transformation from (u, v) to (x, y) is $1/2$ and thus (U, V) is uniform on $\mathcal{G}'$. In particular they are i.i.d. and uniformly distributed on $[-1, 1]$.

Exercise 10

Let the random vector (α, β) be uniformly distributed in the upper semicircle $\mathcal{G} = \{x^2 + y^2 < 1, y > 0\}$. That is, the corresponding two-dimensional probability density is

$$f_{(\alpha,\beta)}(x, y) = \begin{cases} \text{const} & x, y \in \mathcal{G} \\ 0 & x, y \notin \mathcal{G} \end{cases}$$

a. What is the value of the constant in the formula?
b. Find the density $f_\alpha(x)$ of the first coordinate α of the vector.
c. Find the probability density for $\rho = \sqrt{\alpha^2 + \beta^2}$. Draw the graph of $f_\rho(t)$.
d. Find the probability density for $\phi = \arccos(\alpha/\sqrt{\alpha^2 + \beta^2})$. Draw the graph of $f_\phi(t)$.
e. Are ρ and ϕ dependent?

f. Find the probability density for $\xi = \alpha/\beta$. Draw the graph of $f_\xi(t)$.
g. Find the probability density for $\eta = \alpha^2/\beta^2$. Draw the graph of $f_\eta(t)$.
h. Find the probability density for $\theta = \alpha^2 + \beta^2$. Draw the graph of $f_\theta(t)$.

Solution

a. The constant is $1/|\mathcal{G}| = 2/\pi$.
b. The density of α is obtained as pushing forward the uniform measure on $\mathcal{G}$ on the segment $[-1, 1]$. So a graphical visualization immediately shows $f_\alpha(x) = \frac{2}{\pi}\sqrt{1-x^2}\mathbf{1}_{[-1,1]}$. We can also find this by computing $f_\alpha(x) = \int f_{\alpha,\beta}(x, y)dy$.
c. ρ is the polar radius. $F_\rho(r) = \mathbb{P}(\rho \le r) = (\pi r^2/2)/(\pi/2) = r^2$ for $r \in [0, 1]$. So $f_\rho(r) = 2r\mathbf{1}_{[0,1)}$.
d. ϕ is the polar angle. It is uniformly distributed on $[0, \pi]$.
e. In polar coordinates, the joint density is $f(r, \phi) = (2/\pi) \cdot r$. Therefore ρ and ϕ are independent.
f. $\xi = \alpha/\beta = \cot(\phi)$. $F_\xi(x) = \mathbb{P}(\cot\phi \le x) = \mathbb{P}(\phi \ge \operatorname{arccot}(x)) = \frac{\pi - \operatorname{arccot}(x)}{\pi}$. $f_\xi(x) = \frac{1}{\pi(1+x^2)}$ (known as Cauchy distribution).
g. $\eta = \xi^2$. $F_\eta(y) = \mathbb{P}(\xi^2 \le y) = F_\xi(\sqrt{y}) - F_\xi(-\sqrt{y}^-)$. $f_\eta(y) = \frac{1}{\pi\sqrt{y}(1+y)}\mathbf{1}_{[0,\infty)}(y)$.
h. $\theta = \rho^2$. $F_\theta(t) = \mathbb{P}(\rho^2 \le t) = (\sqrt{t})^2 = t$ for $t \in [0, 1]$ (uniform distribution).