# Bell's Inequalities and the Limits of Applicability of Probability Models

The most interesting examples of attempts to use the traditional Kolmogorov probability model in the microworld are the interpretations of the Stern-Gerlach experiment on the deflection of a particle beam in a magnetic field and Alain Aspect's experiment on the interference of photon states (an experiment originally proposed back in the 1930s in the Einstein-Podolsky-Rosen paper). In 1964, a relatively simple result in probability theory appeared, which showed the incompatibility of traditional probability models with the quantitative measurements in these experiments. This result is called *Bell's inequalities for random variables*; a detailed explanation of the connection between the inequalities and physical measurements can be found in Alexander Lvovsky's 2019 textbook 'Excellent Quantum Mechanics'. Below is a proof (due to Accardi) of Bell's inequalities for random variables.

*Remark.* The original proof by Bell and almost all later published proofs of Bell's inequality use only random variables that take only two values, $+1$ and $-1$.

## Arithmetic Inequalities

**Lemma 0.1.** *For any two numbers $a, c \in [-1, 1]$, the following two inequalities (the variant for the signs $+$ and $-$) hold:*

$$|a \pm c| \leqslant 1 \pm ac \tag{1}$$

*Moreover, equality in expression Equation 1 holds if and only if either $a = \pm 1$ or $c = \pm 1$.*

*Lemma 0.1.* The two variants of inequalities Equation 1 follow from the fact that one is obtained from the other by changing the sign of $c$, since $c$ is chosen arbitrarily in $[-1, 1]$. Since for any $a, c \in [-1, 1]$, we have $1 \pm ac \geqslant 0$, Equation 1 is equivalent to $|a \pm c|^2 = a^2 + c^2 \pm 2ac \leqslant (1 \pm ac)^2 = 1 + a^2c^2 \pm 2ac$, and this is equivalent to the inequality $a^2(1 - c^2) + c^2 \leqslant 1$, which holds identically, since $1 - c^2 \geqslant 0$, and, consequently,

$$a^2(1 - c^2) + c^2 \leqslant 1 - c^2 + c^2 = 1 \tag{2}$$

Note that in expression Equation 2, equality holds if and only if $a = \pm 1$ or $c = \pm 1$. Since the inequality in expression Equation 1 remains unchanged when $a$ and $c$ are swapped, the statement follows. □

**Corollary 0.1.** *For any three numbers $a, b, c \in [-1, 1]$, the following equivalent inequalities (for the sign variants $+$ and $-$) hold:*

$$|ab \pm cb| \leqslant 1 \pm ac \tag{3}$$

*and equality holds if and only if $b = \pm 1$ and either $a = \pm 1$ or $c = \pm 1$.*

*Corollary 0.1.* For $b \in [-1, 1]$,

$$|ab \pm cb| = |b| \cdot |a \pm c| \leqslant |a \pm c|$$

Thus, the statement follows from Lemma 0.1, and the first equality holds if and only if $b = \pm 1$, so the second statement also follows from Lemma 0.1. □

**Lemma 0.2.** *For any numbers $a, \tilde{a}, b, \tilde{b}, c \in [-1, 1]$, we have*

$$|ab - cb| + |a\tilde{b} + c\tilde{b}| \leqslant 2 \tag{4}$$

$$ab + a\tilde{b} + \tilde{a}\tilde{b} - \tilde{a}b \leqslant 2 \tag{5}$$

*Equality in the first formula holds if and only if $b, \tilde{b}, a, c = \pm 1$.*

*Lemma 0.2.* From inequality Equation 3

$$|ab - cb| \leqslant 1 - ac \tag{6}$$

$$|a\tilde{b} + c\tilde{b}| \leqslant 1 + ac \tag{7}$$

Adding them up, we get expression Equation 4. The left-hand side of expression Equation 5 is less than or equal to

$$|ab - b\tilde{a}| + |a\tilde{b} + \tilde{b}\tilde{a}|$$

and by replacing $\tilde{a}$ with $c$, expression Equation 7 becomes the left-hand side of expression Equation 4. Conversely, suppose that equality holds in expression Equation 4, and suppose that either $|b| < 1$ or $|\tilde{b}| < 1$. Then we arrive at a contradiction.

$$2 = |b| \cdot |a - c| + |\tilde{b}| \cdot |a + c| < |a - c| + |a + c| \leqslant (1 - ac) + (1 + ac) = 2$$

Thus, if equality holds in expression Equation 4, it must be that $|b| = |\tilde{b}| = 1$. In this case, expression Equation 4 takes the form

$$|a - c| + |a + c| = 2$$

and, if either $|a| < 1$ or $|c| < 1$, then it follows from Lemma 0.1 that $|a - c| + |a + c| < (1 - ac) + (1 + ac) = 2$ so it must also be that $a, c = \pm 1$. □

**Corollary 0.2.** *If $a, \tilde{a}, b, \tilde{b}, c \in \{-1, 1\}$, then the inequalities in expression Equation 3 and expression Equation 4 are equivalent, with equality holding in all of them. However, the inequality in expression Equation 5 can be strict.*

*Corollary 0.2.* We know that the inequalities in expressions Equation 1 and Equation 2 are equivalent; also, Equation 4 follows from Equation 1. Choosing $\tilde{b} = a$ in expression Equation 4, since $a = \pm 1$, expression Equation 4 takes the form $|ab - cb| \leqslant 1 - ac$, which is equivalent to $a(b + \tilde{b}) + \tilde{a}(\tilde{b} - b) \leqslant 2$.

Under our assumptions, either $(b + \tilde{b})$ or $(\tilde{b} - b)$ is zero, so the inequality $a(b + \tilde{b}) + \tilde{a}(\tilde{b} - b) \leqslant 2$ (see Equation 5) is equivalent to either $a(b + \tilde{b}) \leqslant 2$ or $\tilde{a}(\tilde{b} - b) \leqslant 2$, and in both cases, we can choose $a, b, \tilde{b}$ or $\tilde{a}, b, \tilde{b}$ such that the product is negative and the inequality is strict. □

### Bell's Inequalities for Random Variables

**Theorem 0.1** (Bell's Theorem). *Let $(\xi_1, \xi_2, \xi_3, \xi_4)$ be a random vector with components whose absolute values do not exceed 1. Then the following three inequalities hold*

$$\mathbb{E}\Big[|\xi_1\xi_2 - \xi_2\xi_3|\Big] \leqslant 1 - \mathbb{E}[\xi_1\xi_3]$$

$$\mathbb{E}\Big[|\xi_1\xi_2 + \xi_2\xi_3|\Big] \leqslant 1 + \mathbb{E}[\xi_1\xi_3]$$

$$\mathbb{E}\Big[|\xi_1\xi_2 - \xi_2\xi_3|\Big] + \mathbb{E}\Big[|\xi_1\xi_4 + \xi_3\xi_4|\Big] \leqslant 2,$$

*where the first and second inequalities are equivalent. If, however, $\xi_1$ or $\xi_3$ are discrete with values $\pm 1$, then all three inequalities are equivalent.*

*Theorem 0.1*. On the probability space $\Omega$ of the random vector, we use the arithmetic inequalities obtained above pointwise, together with $|E(\alpha)| \leqslant E(|\alpha|)$. $\square$